Hamilton-Jacobi theory

Hamilton-Jacobi theory gives us a general procedure for finding a canonical transformation so that the Hamilton's equations of motion for the transformed Hamiltonian are trivially solvable.

We require the transformed Hamiltonian K to be identically zero:

$$K = 0 \tag{1}$$

so that the new variables Q, P are constants in time:

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0$$
$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0 \tag{2}$$

For the old Hamiltonian H and the new one K holds:

$$K = H + \frac{\partial F}{\partial t} \tag{3}$$

where F is the generating function of the canonical transformation. Since $K=0$ eq (3) becomes:

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0 \tag{4}$$

Let us consider a generating function $F_2(q, P, t)$ which is a function of the old coordinates q_i, and the new constant momenta P_i. We denote this function as $S(q, P, t)$.

The equations of the transformation give:

$$p_i = \frac{\partial S}{\partial q_i} \qquad i = 1, 2, \dots, 2n \tag{5}$$

so that eq. (4) becomes:

$$\boxed{H\left(q_1, \dots, q_n; \frac{\partial S}{\partial q_1}, \dots \frac{\partial S}{\partial q_n}; t\right) + \frac{\partial S}{\partial t} = 0} \tag{6}$$

This equation, known as <u>the Hamilton-Jacobi equation</u> is a partial differential equation in $(n+1)$ variables $q_1, q_2, \dots, q_n; t$ for the generating function S which is called <u>Hamilton's principal function</u>.

Suppose there exists a solution of eq. (6) of the form

$$S = S(q_1, \dots, q_n; \alpha_1, \dots, \alpha_{n+1}; t) \tag{7}$$

where the quantities $\alpha_1, \dots, \alpha_{n+1}$ are $n+1$ independent constants of integration. Since S itself does not appear in eq. (6), but only its partial derivatives with respect to q or t are involved, one of the $n+1$ constant $\alpha_1, \dots \alpha_{n+1}$ must appear only as an additive constant in S.

So for our purposes S can be written as:

$$S = S(q_1, \ldots, q_n; \alpha_1, \quad \alpha_n; t) \tag{8}$$

where none of the n independent constants is solely additive.

We can take the n constants of integration to be the new <u>constant</u> momenta:

$$P_i = \alpha_i \qquad i = 1, \ldots, n \tag{9}$$

so that

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \tag{10}$$

$$Q_i = b_i = \frac{\partial S(q, \alpha, t)}{\partial a_i} \tag{11}$$

where b_i are constant obtained from the initial conditions.

If the old Hamiltonian does not depend on time explicitly the above procedure is simplified. The Hamilton-Jacobi equation (eq. (6)) becomes:

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}\right) = 0 \qquad (12)$$

The first term involves only the dependence on t, whereas the second term conserned only with the dependence of S on the q_i. The time variable can therefore be separated by assuming a solution for S of the form:

$$\boxed{S(q_i, \alpha_i, t) = W(q_i, a_i) - \alpha_1 \cdot t} \qquad (13)$$

So the Hamilton-Jacobi equation becomes:

$$\boxed{H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1} \qquad (14)$$

It can be shown that W generates a canonical transformation in which the new momenta P_i are all constants of the motion α_i, where a_1 in particular is the constant of motion H.

The generating function W is called <u>Hamilton's characteristic function</u>

The equations of the transformation are:

$$p_i = \frac{\partial W}{\partial q_i} \quad , \quad Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i} \tag{15}$$

and the new Hamiltonian

$$K = H + \underbrace{\frac{\partial W}{\partial t}}_{=0} \Longrightarrow K = H \overset{(14)}{=\!=} \alpha_1 \tag{16}$$

The Hamilton's equations of motion give:

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \Longrightarrow P_i = \alpha_i \quad i=1,\dots n \tag{17}$$

$$\dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = \begin{cases} 1 & i=1 \\ 0 & i\neq 1 \end{cases} \Longrightarrow$$

$$\begin{cases} Q_1 = t + b_1 \equiv \dfrac{\partial W}{\partial \alpha_1} \\ Q_i = b_i \equiv \dfrac{\partial W}{\partial \alpha_i} & i \neq 1 \end{cases} \tag{18}$$

<u>NOTE</u>: Instead of α_1 in (14) we could have a function of all α_i's:

$$K = K(\alpha_1, \alpha_2, \quad \alpha_n)$$

Then the equations of motion become:

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \Longrightarrow P_i = \alpha_i \quad i=1,\dots,n \tag{17b}$$

$$\dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = v_i(\alpha_i) = \text{constant} \Longrightarrow$$

$$Q_i = v_i \cdot t + b_i \tag{18b}$$

Example: The harmonic oscillator

The Hamiltonian is:

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \tag{19}$$

where

$$\omega = \sqrt{k/m}$$

By setting $p = \frac{\partial S}{\partial q}$ the Hamilton-Jacobi equation is:

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \tag{20}$$

Let:

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t \tag{21}$$

Then eq.(20) becomes:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha \tag{22}$$

which integrates to:

$$W = \sqrt{2m\alpha}\int\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}\,dq \tag{23}$$

so that:

$$S = \sqrt{2m\alpha}\int\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}\,dq - \alpha t \tag{24}$$

The solution for q arises from:

$$\beta = \frac{\partial S}{\partial \alpha} \overset{(24)}{=} \sqrt{\frac{m}{2\alpha}} \int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t \tag{25}$$

which gives:

$$t + \beta = \frac{1}{\omega} \arcsin\left(q\sqrt{\frac{m\omega^2}{2\alpha}}\right) \tag{26}$$

or

$$\boxed{q(t) = \sqrt{\frac{2\alpha}{m\omega^2}} \sin[\omega(t+\beta)]} \tag{27}$$

which is the known solution of a harmonic oscillator. For the momentum $p(t)$ we get:

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m\omega^2 q^2} \overset{(27)}{=}$$

$$\sqrt{2m\alpha\left(1 - \sin^2[\omega(t+\beta)]\right)} \Longrightarrow$$

$$\boxed{p(t) = \sqrt{2m\alpha} \cos[\omega(t+\beta)]} \tag{28}$$

The constants α and β are connected with the initial conditions p_0, q_0 for $t=0$ through the relations

$$2m\alpha = p_0^2 + m^2\omega^2 q_0^2 \tag{29}$$

$$\tan \omega\beta = m\omega \frac{q_0}{p_0} \tag{30}$$

Summary

$H(q,p,t)$	$H(q,p) = \text{constant}$

We find a canonical transformation such that

$Q_i, P_i = \text{constants}$	$P_i = \text{constants}$

The new Hamiltonian K is:

$K = 0$	$K = H(P_i) = \alpha_1$

The new equations of motion give:

$\dot{Q}_i = \frac{\partial K}{P_i} = 0 \Rightarrow Q_i = \beta_i$	$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \nu_i \Rightarrow Q_i = \nu_i t + \beta_i$
$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \Rightarrow P_i = \gamma_i$	$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \Rightarrow P_i = \gamma_i$

The generating function is Hamilton's:

Principal Function	Characteristic Function
$S(q, P, t)$	$W(q, P)$

that satisfying Hamilton's-Jacobi equation:

$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$	$H(q, \frac{\partial W}{\partial q}) - \alpha_1 = 0$

Separation of variables in the Hamilton-Jacobi equation

Using the Hamilton-Jacobi theory instead of solving $2n$ ordinary differential equations that make up the canonical equations of motion one must solve a partial differential equation (the Hamilton-Jacobi equation) which in principle is not an easier task.

Under certain conditions, however, it is possible to separate the variables in the Hamilton-Jacobi equation, and the solution can then always be reduced to quadratures.

A coordinate q_j is said to be separable in the Hamilton-Jacobi equation when Hamilton's principal function can be split into two additive parts, one of which depends only on the coordinate q_j and the other is entirely independent of q_j. So if q_j is q_1 we have:

$$S(q_1, \dots, q_n; a_1 \dots a_n; t) = S_1(q_1; a_1 \dots a_n; t) + S'(q_2, \dots, q_n; a_1 \dots a_n; t) \quad (31)$$

and the Hamilton-Jacobi equation can be split into two equations - one separately for S_1 and the other for S'.

The Hamilton-Jacobi equation is completely separable if all the coordinates in the problem are separable

In general a coordinate q_j can be separated if q_j and the conjugate momentum p_j can be segregated in the Hamiltonian into some function $H_j(q_j, p_j)$ that does not contain any of the other variables.

If for example the Hamiltonian has the form:

$$H(q,p,t) = H_1(q_1, p_1) + H'(q_2, \dots q_n, p_2, \dots p_n, t) \qquad (32)$$

then q_1 is separable. A solution for Hamilton's principal function S of the form:

$$S(q_1, \dots q_n; a_1, \dots, a_n; t) = S_1(q_1; a_1 \dots, a_n; t) + S'(q_2, \dots, q_n; a_1 \dots, a_n; t) \qquad (31)$$

splits the Hamilton-Jacobi equation into two parts:

$$H_1\left(q_1, \frac{\partial S_1}{\partial q_1}\right) = c$$

$$H'\left(q_2, \dots, q_n, \frac{\partial S'}{\partial q_2}, \dots, \frac{\partial S'}{\partial q_n}, t\right) = -c \qquad (33)$$

where c=constant

If the Hamilton-Jacobi equation is completely separable a solution for Hamilton's principal function of the form:

$$\underline{S = \sum_i S_i(q_i; \alpha_1, \dots, \alpha_n; t)} \qquad (34)$$

splits the Hamilton-Jacobi equation into n equations of the form:

$$\underline{H_i\left(q_i, \frac{\partial S_i}{\partial q_i}; \alpha_1 \dots, \alpha_n\right) = \alpha_i} \qquad (35)$$

Each of the above equations involves only one of the coordinates q_i and the corresponding partial derivative of S_i with respect to q_i. They are therefore a set of ordinary differential equations easily solved.

Example

Motion of a point mass ($m=1$) on the xy-plane under the influence of the potential

$$V(x,y) = k^2 x y \tag{36}$$

where k = constant.

We have the Hamiltonian:

$$H(x,y) = \frac{1}{2}(p_x^2 + p_y^2) + k^2 x y \tag{37}$$

where

$$p_x = \dot{x} \quad , \quad p_y = \dot{y} \tag{38}$$

NOTE: The separability of the Hamilton-Jacobi equation depends not only on the physical problem involved but also on the choice of the system of the generalized coordinates used.

The x, y coordinates in (37) are not separable. By applying the transformation:

$$v = \frac{1}{\sqrt{2}}(x-y) \qquad x = \frac{1}{\sqrt{2}}(u+v)$$
$$u = \frac{1}{\sqrt{2}}(x+y) \qquad y = \frac{1}{\sqrt{2}}(u-v) \tag{39}$$

the Hamiltonian becomes:

$$H(u,v) = \frac{1}{2}(p_u^2 + k^2 u^2) + \frac{1}{2}(p_v^2 - k^2 v^2) \tag{40}$$

where

$$p_u = \dot{u} \quad , \quad p_v = \dot{v} \tag{41}$$

So the u, v coordinates in (40) are separable

We look for a Hamilton's characteristic function W of the form:

$$W(u, v, \alpha_1, \alpha_2) = W_1(u, \alpha_1, \alpha_2) + W_2(v, \alpha_1, \alpha_2) \qquad (42)$$

The Hamilton-Jacobi equation splits into 2 equations:

$$\left.\begin{aligned} \left(\frac{dW_1}{du}\right)^2 + k^2 u^2 &= 2(\alpha_1 - \alpha_2) \\ \left(\frac{dW_2}{dv}\right)^2 - k^2 v^2 &= 2\alpha_2 \end{aligned}\right\} \quad (43)$$

where $\alpha_1 = h$, the constant value of the Hamiltonian. By solving eqs. (43) we find W as:

$$W = \int [2(\alpha_1 - \alpha_2) - k^2 u^2]^{1/2}\, du + \int [2\alpha_2 + k^2 v^2]^{1/2}\, dv \qquad (44)$$

Using eqs. (18) we find the equations of motion for u and v:

$$\left.\begin{aligned} t - \beta_1 &= \frac{\partial W}{\partial \alpha_1} = \int [2(\alpha_1 - \alpha_2) - k^2 u^2]^{-1/2}\, du \\ \beta_2 &= \frac{\partial W}{\partial \alpha_2} = -(t - \beta_1) + \int [2\alpha_2 + k^2 v^2]^{-1/2}\, dv \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} u &= \frac{\sqrt{2(\alpha_1 - \alpha_2)}}{k} \cdot \sin[k(t - \beta_1)] \\ v &= \frac{\sqrt{2\alpha_2}}{k} \cdot \sinh[k(t - \beta_1 + \beta_2)] \end{aligned}\right\} \quad (45)$$

where $\alpha_1 = h$, α_2, β_1, β_2 are constants defined by the initial conditions.

Using the transformation (39) and eqs. (45) we can find the equations of motion for x and y.

Integrable systems

Let us consider an autonomous Hamiltonian system:

$$H(q_i, p_i) \qquad i = 1, 2, \ldots n \tag{46}$$

We call the Hamiltonian system (46) completely integrable if it possesses n constants of motion

$$I_i(q_i, p_i) = I_i = \text{constant} \tag{47}$$

(which are called integrals), which are:

i) analytic in q_i, p_i
ii) single valued
iii) global (they are constant along every trajectory $p(t), q(t)$ fol all t)
iv) functionally independend of each other
v) 'in involution', i.e. their mutual Poisson brackets vanish:

$$[I_i, I_j] = 0 \quad \forall\, i, j \tag{48}$$

The Hamiltonian system (46) can be solved via the introduction of a canonical transformation such that the n integrals of motion become the new generalized momenta:

$$P_i = I_i(q_k, p_k) \quad k = 1, n \tag{49}$$

assuming that

$$\det \left| \frac{\partial I_i}{\partial p_j} \right| \neq 0 \tag{50}$$

we can solve eqs. (49) and get:

$$p_i = f_i(q_K, P_K) \tag{51}$$

We assume that there exists a suitable generating function

$$S = S(q_i, P_i) \tag{52}$$

which introduce the above-mentioned transformation. Then this generating function satisfies:

$$p_i = \frac{\partial S}{\partial q_i} = f_i(q_j, P_j) \tag{53}$$

It can be proven that

$$S = \sum_i \int f_i(q_j, P_j)\, dq_i \tag{54}$$

So the equations that define the canonical transformation are eqs. (53) and:

$$Q_i = \frac{\partial S}{\partial P_i} = Q_i(q_j, P_j) = Q_i(q_j, I_j) \tag{55}$$

The Hamilton's equations of motion for the new Hamiltonian H' give:

$$\left.\begin{aligned} \dot{Q}_i &= \frac{\partial H'}{\partial P_i} = \frac{\partial H'}{\partial I_i} \\ \dot{P}_i &= \dot{I}_i = -\frac{\partial H'}{\partial Q_i} \end{aligned}\right\} \tag{56}$$

Since I_i = constant we get from (56)

$$\dot{I}_i = 0 \Rightarrow \frac{\partial H'}{\partial Q_i} = 0 \qquad (57)$$

So the new coordinates are cyclic. This means that the new Hamiltonian H' is a function of only the momenta I_i:

$$H' = H'(I_i) \quad i = 1, 2, \dots n \qquad (58)$$

Eqs. (56) give

$$\dot{Q}_i = v_i = \frac{\partial H}{\partial I_i} = \text{constant} \qquad (59)$$

since v_i is a function of I_i which are constants
Thus the equations of motion (56) give

$$P_i = I_i = \text{constant} \quad (\text{"Action" variable}) \qquad (60)$$

$$Q_i = v_i(I_i) \cdot t + \beta_i \quad (\text{"Angle" variable}) \qquad (61)$$

where v_i, β_i are constants.

<u>Liouville - Poincaré theorem:</u>
A completely integrable Hamiltonian system can be solved by simple integrations!

Integrability $\Longrightarrow$ Solvability

Action - Angle Variables

- They are canonically conjugate variables
- They exist ONLY FOR INTEGRABLE SYSTEMS: n-degrees of freedom systems possessing n integrals

We consider a Hamiltonian H which does not explicitly depend on t:

$$H(q_i, p_i) \quad i=1,2,\dots,n \tag{62}$$

Then Hamilton's Characteristic Function $W(q_i, I_i)$ generates a canonical transformation

$$(q,p) \longrightarrow (\theta, I) \tag{63}$$

such that:

$$p_i = \frac{\partial W}{\partial q_i} \quad , \quad \theta_i = \frac{\partial W}{\partial I_i} \tag{64}$$

The Action-Angle variables are convenient for coordinating the motion on the invariant tori of integrable systems.
An n-dimensional torus M possess n irreducible circuits γ_k which cannot be "shrunk" continuously to zero. $W(q, I)$ changes by a nonzero quantity everytime we go around one of these circuits

The Action variable I_k is defined as this change of W around the irreducible circuit γ_k

$$I_k = \frac{1}{2\pi} \oint_{\gamma_k} \sum_{i=1}^{n} p_i \, dq_i \qquad (65)$$

so $I_k = \frac{1}{2\pi} \times$ (areas of projections of γ_k on the planes $p_1 q_1, p_2 q_2, \ldots, p_n q_n$)

Using (64) we get:

$$I_k = \frac{\Delta W_k}{2\pi} \qquad (66)$$

where ΔW_k is the change of W for the kth circuit of the torus M.

So I_k's define the torus M.

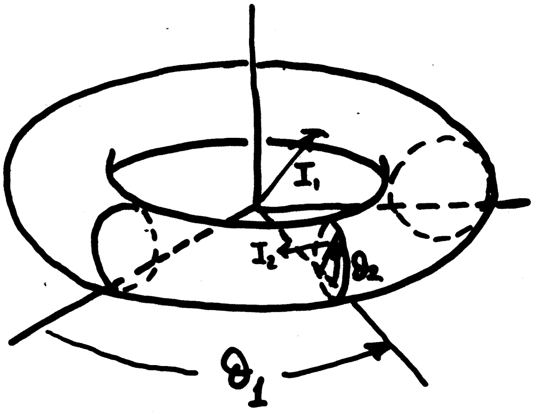
The motion <u>ON</u> the torus M is coordinated by the n angle variables ϑ_k canonically conjugated to I_k's given by eq. (64):

$$\vartheta_k = \frac{\partial W}{\partial I_k}(q, I) \qquad (67)$$

We call the variables ϑ_k angles because as we traverse circuit γ_k, ϑ_k changes by 2π while all other ϑ's remain constant:

$$(\Delta\vartheta_\ell)_{\gamma_k} \overset{(67)}{=} \Delta_{\gamma_k}\left(\frac{\partial W}{\partial I_\ell}(q, I)\right) = \frac{\partial}{\partial I_\ell} \Delta_{\gamma_k} W \overset{(66)}{=}$$

$$= \frac{\partial}{\partial I_\ell}(2\pi I_k) = 2\pi \delta_{\ell k}$$



So with the canonical transformation

$$(q, p) \longrightarrow (\vartheta, I) \qquad (63)$$

the variables q, p are periodic in ϑ with period 2π i.e:

$$q_k = \sum_{\vec{n}} A_{\vec{n}}^{(k)}(I) \cdot e^{i(\vec{n}, \vec{\vartheta})}$$

$$p_k = \sum_{\vec{n}} B_{\vec{n}}^{(k)}(I) \cdot e^{i(\vec{n}, \vec{\vartheta})} \quad k=1,2,\dots,n \qquad (68)$$

where the summation proceed over all integer vectors $\vec{n} \equiv (n_1, n_2, \dots, n_n)$ and

$$(\vec{n}, \vec{\vartheta}) \equiv n_1 \vartheta_1 + n_2 \vartheta_2 + \dots + n_n \vartheta_n \qquad (69)$$

The angles are given be eq. (61):

$$\vartheta_k = \nu_k(I)\, t + \delta_k \qquad (70)$$

where:

$$\nu_k(I) = \frac{\partial H'}{\partial I_k} \qquad (71)$$

so the quantities ν_k will be referred to, as the n frequencies of the problem.

NOTE: Some authors use as a definition of the Action variables:

$$I_k = \frac{1}{2\pi} \oint p_k \, dq_k \qquad (72)$$

where the integration is performed over a complete period of the q_k variable. This definition is restricted to q, p variables in which the system is already separated into n uncoupled subsystems, eg a set of n uncoupled harmonic oscillators.

To see this note that the angle θ_k changes by 2π over a complete period of q_k but is unaffected by a similar change in q_ℓ, $\ell \neq k$:

$$(\Delta\theta_k)_\ell = \oint_\ell \frac{\partial \theta_k}{\partial q_\ell} \, dq_\ell \qquad (73)$$

where the change is taken over a period of q_ℓ. Using (67) in (73) we find:

$$(\Delta\theta_k)_\ell = \oint_\ell \frac{\partial^2 W}{\partial q_\ell \, \partial I_k} \, dq_\ell = \frac{\partial}{\partial I_k}\left(\oint_\ell \frac{\partial W}{\partial q_\ell} \, dq_\ell \right) \stackrel{(64)}{=}$$

$$= \frac{\partial}{\partial I_k} \oint_\ell p_\ell \, dq_\ell \stackrel{(72)}{=} \frac{\partial}{\partial I_k} (2\pi I_\ell) = 2\pi \delta_{\ell,k}$$

Thus we see now combining this result with (70) that $\nu_k(I)$ can be identified with the frequency of oscillation of the q_k variable.

Hence the solutions $q_k(t)$, $p_k(t)$ can be written as single sum Fourier series:

$$q_k(t) = \sum_{n=-\infty}^{\infty} A_n^{(k)}(I) \cdot e^{in(\nu_k \cdot t + \beta_k)}$$

$$k = 1, 2 \dots n \qquad (74)$$

$$p_k(I) = \sum_{n=-\infty}^{\infty} B_n^{(k)}(I) \cdot e^{in(\nu_k t + \beta_k)}$$

clearly indicating that the system is completely separable in the q, p variables.

The solutions (68) are more general than (74) because the definition (65) does not assume that the system is separable in q, p variables.

If it is separable in p, q then (65) reduces to

$$I_k = \frac{1}{2\pi} \int_{\gamma_k} p_k \, dq_k \quad \text{(separable in } q, p) \qquad (75)$$

which is analogous to definition (72).

Example: Two uncoupled harmonic oscillators

The Hamiltonian of the system is:

$$H = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) \qquad (76)$$

This Hamiltonian has two integrals:

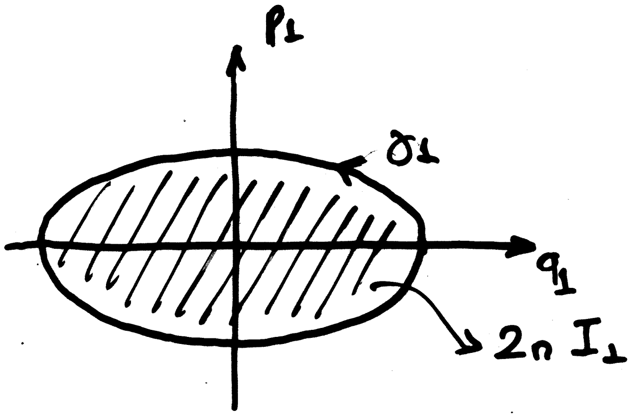
$$F_1 = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2), \quad F_2 = \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) \qquad (77)$$

$$I_1 = \frac{1}{2\pi}\oint_{\gamma_1} p_1\, dq_1 =$$

$$= \frac{1}{2\pi}\int \sqrt{2(F_1 - \omega_1^2 q_1^2)}\, dq_1 =$$

$$= \frac{1}{2\pi} \times (\text{area of ellipse}) =$$

$$= \frac{\sqrt{2F_1}}{2}\sqrt{\frac{2F_1}{\omega_1^2}} = \frac{F_1}{\omega_1}$$



So we get:

$$I_1 = \frac{F_1}{\omega_1}, \quad I_2 = \frac{F_2}{\omega_2} \qquad (78)$$

The Hamiltonian in Action variables is

$$H = F_1 + F_2 = \omega_1 I_1 + \omega_2 I_2 = H'(I_1, I_2) \qquad (79)$$

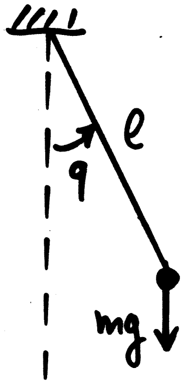
The frequencies of oscillation ν_1, ν_2

$$\nu_1 = \frac{\partial H'}{\partial I_1} = \omega_1, \quad \nu_2 = \frac{\partial H'}{\partial I_2} = \omega_2 \qquad (80)$$

are identical with the harmonic frequencies as is expected in a linear system.

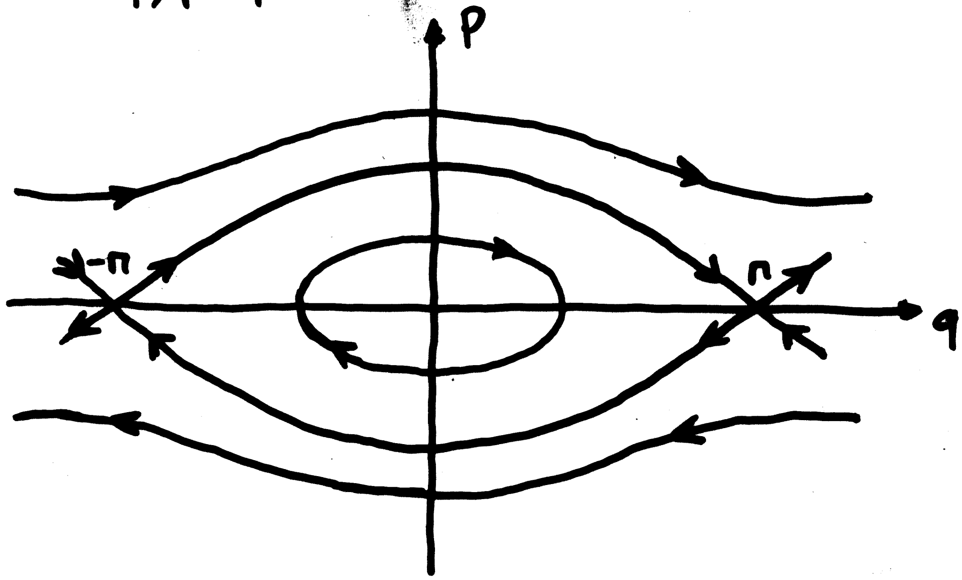
Example : The pendulum

We consider a non-linear system in one dimension: a simple pendulum of length ℓ and mass m. Its Hamiltonian is:



$$H(q,p) = \frac{1}{2}\frac{p^2}{m\ell^2} - mg\ell\cos q = E \tag{81}$$

The manifold $\mathcal{M}$ is one-dimensional: a family of closed curves on the q, p plane:



The Action for $E > mg\ell$ is:

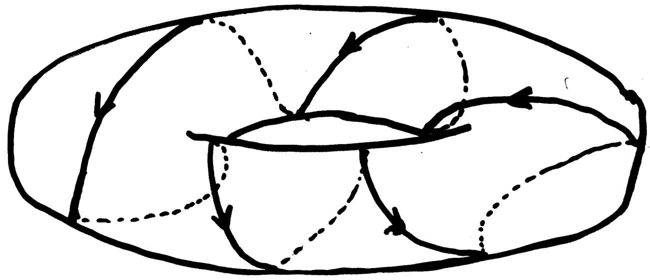
$$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{\pi}\int_0^{\pi}\sqrt{2m\ell^2(E + mg\ell\cos q)}\,dq \tag{82}$$

Equation (82) defines implicitly the new Hamiltonian $H' = H'(I) = E$ as a function of solely the Action variable I. So the frequency of oscillation of the pendulum is a function of I given by:

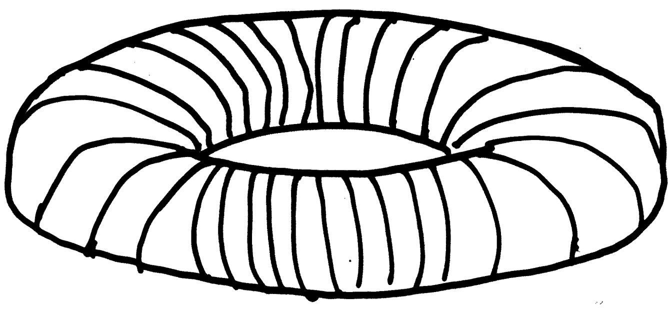
$$\nu(I) = \frac{\partial H'(I)}{\partial I} \tag{83}$$

Motion on the torus

The general solutions of integrable systems given by equations (68) and (69), descibe orbits which lie on an M-dimensional torus. These orbits either close upon themselves at some point (periodic orbit)



or never close upon themselves and cover the torus densely (quasiperiodic orbits)



What is the requirement on the frequencies v_K so that the orbit will be periodic?

They must all be integer multiples of the overall frequency of the oscillation which we call 'recurrence' frequency v_r, i. e.

$$v_K = m_K v_r \qquad k=1,2,..n \tag{84}$$

for a set of positive integers $m_1, m_2, \ldots, m_n$.

So closure of orbits occurs after m_1 circuits of ϑ_1, m_2 circuits of ϑ_2, ..., m_n circuits of ϑ_n

Is closure the rule?

No it is the exception. Think of $N=2$, a two degrees of freedom case. Closure of the orbits would imply from (84) that the two frequencies of the problem ν_1, ν_2 are rationally dependent

$$\frac{\nu_1}{\nu_2} = \frac{m_1}{m_2} = \text{rational} \qquad (85)$$

We know that rationals are dense in all reals but nevertheless form a set of measure zero.

Thus we conclude by saying that all solutions of integrable systems of n degrees of freedom lie on n-dimensional tori. These tori are generically covered by quasiperiodic orbits but there is also a set of tori (of measure zero) on which all orbits close upon themselves, i.e are periodic.